C[ontinuations]:

continued

and

to be continued

At 関数型まつり 2025 June 14, 2025

Satoru Kawahara

Twitter絵師ハカセの質問コーナー [1]

Q. What is the most expressive concept of **functional programming**?

[1] https://arkw.net/products/web/hakase/

[*1]

Q. What is the most expressive concept of **functional programming**?

A. Definitely, it's a

CONTINUATION!

[要出典]

[*1] https://arkw.net/products/web/hakase/

Today's Topic

Learn about

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Learn about

Continuations:

The Concept, History, and Practice

Continuations:
continued and to be continued

- **1. Introduction**
  - Today’s Topic
  - Continuations?
- **2. The Evolution of Continuations**
  - Dump and J operator, SECD machine
  - `call/cc`: `goto` Practical
  - Delimited Continuations
  - Continuation-Passing Style
  - Compiling with Continuations
  - Correspondence between procedural
- **3. Continuations with Effects**
  - Why continuations for effects?
  - Monads
  - Algebraic Effect Handlers
- **4. Conclusion**

Who talks

- 🏢 **eiicon, co.,ltd.**
- 📎 Nymphium
- ❤️ OCaml
- 🤓 Interested in:
  - Programming language theory and implementations
  - Control flow and its operators

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- 💪 Motto:

継続は力なり

Continuation is power

こんにちは、
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Continuations?

Do you *remember* continuations?

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🙋‍♂️ Hmm. . . . . . I forget. Help me remember!

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- 👴 I'm one with continuations.

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- 🙋‍♂️ Hmm. . . . . . I forget. Help me remember!
- 🙋‍♂️ ✅ **Yeah, I think they're callback functions.**
- 👴 I’m one with continuations.

Yes, they are just
callback functions!

●Continuations?

E.g. Read a file, then pass the result to a callback

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readFile(file).flatMap { data ⇒ ...... }
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((v3) ⇒   ......  )))
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def x ▷ k ≡ k(x)
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Pseudo Continuation-Passing Style Conversion

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Continuations are . . . . . .

The Rest of The Computation!!

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def x ▷ k ≡ k(x)
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Continuations:
continued and to be continued

Dump and J operator, SECD machine

The first abstract machine for evaluating functional programs[4]:

`Stack`	:	Stores intermediate results
`Environment`	:	Stores variables
`Control`	:	Stores the next instruction
`Dump`	:	Stores the suspended computation

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This marks the origin of **continuations!**[17][7]

call/cc: goto Practical

Operators in Scheme[16], notably `call/cc`[15][*2]:
Captures the current continuation as a first-class value

```
(let {[(x (call/cc (λ (K)
                     (+ 2 (K 4)))))]}
    (+ x 3))
; ⇒ (let {[x 4]} (+ x 3))
; ⇒ returns 7
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👉 **ANYTHING related to *control transfer***

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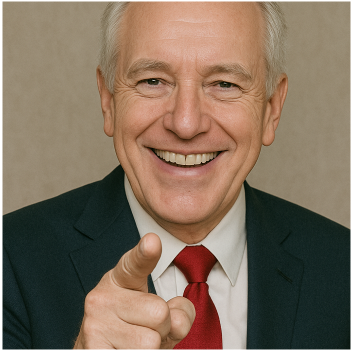
Solution:

Use
Delimited Continuations! [9]

- ✅ capturing ~~*scoped*~~ delimited continuations
- ✅ more structured and composable



Delimited Continuations

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- fcontrol/run[14]
- *multiprompt* extensions[8]
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Delimited continuations enable us to implement

✅ `call/cc`!

✅ **ALL monads!!**

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Delimited continuations enable us to implement

✅ call/cc!

✅ **ALL monads!!**

…… and **vice-versa!**[8]

call/cc

delimited controls ⟷ monads

Continuation-Passing Style

A program representation where control flow is made *explicit* by chaining computations as continuations[12]:

```
(define (add1 x)
  (+ x 1))
(define (mul2 x)
  (* x 2))

(mul2 (add1 3))
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CPS Conversion!

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(define (add1 x κ)
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(add1 3 (λ (smu)
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CPS *fixes* the order of evaluation and control flow, so that it's a good choice for

an intermediate representation

for language implementations!

Compiling with Continuations

CPS as an intermediate representation (IR) for language impls[1]:

$$\boxed{\text{Source}} \longrightarrow \boxed{\text{CPS}} \xrightarrow[\text{(Opts.)}]{} \boxed{\text{Codegen}}$$

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Scheme
OCaml
(flambda wasm)

Source → CPS → (Opts.) → Codegen

✅ Good for **functional languages**

CPS operates functions as first-class values!

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Constant Folding & Inlining	⇒ β reduction
Defunctionalization	⇒ η reduction
Common Subexpression Elimination	⇒ *EZ*:

```
let z = a * b + a * b
in e
```

same!

CPS Conversion ($[\![-]\!]$)

```
( *' ) a b (fun v₁ ->
( *' ) a b (fun v₂ ->
( +' ) v₁ v₂ (fun z ->
⟦e⟧
```

✅ Good for **functional languages**

CPS operates functions as first-class values!

✅ Good for **optimizations**

Several optimizations can be done by β/η,
and each values are single-assignment!

Correspondence between procedural

How do *control transfers* correspond in functional and procedural?

- Functional side:

- Procedural side:

---

*3 Static Single-Assignment form

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- Functional side:
  - ⬇️ J: Low-level continuation operator
- Procedural side:
  - ⬇️ `jmp`: Low-level control transfer

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- Functional side:
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  - 👉 `for` / `while` / `if`: Structured, clear control[5]

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  - ⚙️ **CPS as an IR**
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  - 👉 `for` / `while` / `if`: Structured, clear control[5]
  - ⚙️ **SSA**[*3] **as an IR**

And also,
CPS $\Leftrightarrow$ **SSA!** [2]

[*3] Static Single-Assignment form

Continuations:

continued and to be continued

Why continuations for effects?

Effects, side effects or computational effects:

- Exception
- Async I/O
- Coroutines
- etc.

Handling effects is about what happens——

Why continuations for effects?

Effects, side effects or computational effects:

- Exception . . . . *halt?*
- Async I/O . . . . *when to resume?*
- Coroutines . . . . *which order to resume?*
- etc.

Handling effects is about what happens——

and *when* and *how* to **resume**.

●Why continuations for effects?

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- Exception .... *halt?*
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- etc.

Handling effects is about what happens——
and *when* and *how* to resume.

So, it's time for

Continuations!

Monads

An approach to modeling calculi with *effects*[11]:

```
class Monad m where
  return :: a -> m a
  (>>=)  :: m a -> (a -> m b) -> m b
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instance Monad Maybe where
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  Just x  >>= k = k x
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throw away continuation
to stop computation!

```
maybe e >>= \x -> u
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⇓ direct-style with do

```
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  u
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Algebraic Effect Handlers

A new way to model effectful computations, with high *modularity* and *composablility*[3]:

```
type _ eff += Print : string -> unit eff

match perform @@ Print "hello" with
| effect (Print msg),  k ->
  print_endline msg;  continue k ()
```

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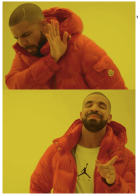
Handlers make it *easier* to **compose** and **modularise** than monad transformers!


[*a]

Monad Transformers

Algebraic Effect Handlers

[*a] https://www.youtube.com/watch?v=uxpDa-c-4Mc

Conclusion

Summary

- Explain from the history to recent trends of **continuations**
- Continuations are powerful, joyful, and useful!
- Research about continuations is **still hot topic**, and **continued!**

Couldn't talk today 😭

- Type Systems
  - Answer-Type Modification
  - Linear Types for efficient runtime repr
  - Type-preserving conversion
- Formal Semantics for Natural Languages
- Recursion
- ······ and *anything* about computation with continuations.

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